THE OPTIMAL GUARANTEEING CONTROL WITH SINGLE SWITCHING

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A. P. CHERENKOV
(Moscow)
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A system subject to small perturbations is investigated. The motion of the system consists of two successive stages. The criterion of the qualitv of system operation is the value of a certain functional of its phase trajectory. The first stage of motion is succeeded by the second stage when a certain quantity dependent on the time, perturbations, and controlling parameters reaches a certain value. A typical example of such a system is a rocket for which the transition from powered flight to coasting occurs at the instant of attainment of a programmed value of some quantity measured during flight.

The necessary and sufficient condition for the existence of a control ensuring the invariance of the system with respect to perturbations is formulated. The optimal guaranteeing control is determined for cases where the domain of possible values of the perturbations is an n-dimensional ellipsoid or parallelepiped with its center at the origin.

1. Let us consider a system Λ whose motion takes place in two distinct successive stages. The system Λ is acted on by small perturbations defined by the n-dimensional vector ε. The functional V of the phase trajectory of the system Λ has been defined. The symbol d denotes the principal part of the deviation from the unperturbed value ; this principal part is linear in ε . We take the quantity $|dV|$ as our criterion of system operation quality: the smaller the value of this quantity, the better. Transition from the first stage of motion to the second occurs upon attainment of a certain value of the quantity η. This quantity is a linear form (switching function) of s quantities constituting a known vector function of the time and perturbations; $u = u(t, \varepsilon)$. Control consists in selecting the vector φ of coefficients of the switching function $\eta = (\varphi, u)$.

An important practical application of this method of control is that of reducing rocket scatter by suitable adjustment of the instant of thrust cutoff. The various aspects of such control have been dealt with by many authors (e. g. see [1, 2]).

Our paper [3] concerns the stochastic variant of the problem in which the stochastic characteristics ε are assumed known and it is necessary to find the control which minimizes the dispersion of the quantity dV. In the present paper we assume knowledge of only the domain E of possible values of ε . The optimal control is defined as that control which minimizes the maximum quantity $|dV|$ possible for $\varepsilon \in \mathrm{E}$. The necessary and sufficient condition for the equality of this minimum to zero is formulated. The optimal control for those cases where E is an ellipsoid or parallelepiped with its center at the origin is determined.

2. Let us assume that

$$dV = (M, \varepsilon) + \omega d\tau, \qquad du = L\varepsilon + Hd\tau$$

in the neighborhood of $\varepsilon = 0$ and of the unperturbed value of the instant of switching τ.

Here M, ω, L, H are known matrices of the orders $n \times 1$, 1×1, $s \times n$, $s \times 1$, respectively. Since the equation $d\eta = 0$ is fulfilled to within higher-order terms by virtue of the switching condition, we can write

$$dV = \frac{\varepsilon' S\varphi}{(H, \varphi)} \qquad (S = MH' - \omega L')$$

We see that control by the above method with small perturbations is possible if and only if $(H, \varphi) \neq 0$. We shall assume that the inequality $H \neq 0$ required for this to be the case is always fulfilled. Since multiplication of the vector φ by a nonzero constant does not alter the quantity dV, we shall consider only the vectors φ belonging to the hyperplane $\Phi = \{\varphi : (H, \varphi) = 1\}$. By virtue of this normalization of the switching function we have

$$dV = \varepsilon' S\varphi$$

Our problem consists in finding

$$\mu_0 = \inf_{\varphi \in \Phi} \sup_{\varepsilon \in E} |\varepsilon' S\varphi| \tag{2.1}$$

A problem similar to (2.1) is investigated in [4]. The present study differs from [4] in the fact that the author of the latter paper uses a numerical method of solution and considers a broader class of domains E.

3. Let us cite some of the results of our paper [3], which concerns a stochastic variant of the problem. Let

$$\langle \varepsilon \rangle = 0, \qquad \langle \varepsilon\varepsilon' \rangle = D$$

where D is a known correlation matrix.

$$\langle (dV)^2 \rangle = \varphi' B\varphi \qquad (B = A'A, \quad A = \sqrt{D}S)$$

(the symbol $\sqrt{\ }$ denotes the positive square root [5]). The quantity $\lambda = \min \langle (dV)^2 \rangle$ under the condition $\varphi \in \Phi$ and the optimal vector φ can be determined from the system of equations

$$B\varphi = \lambda H, \qquad (H, \varphi) = 1 \tag{3.1}$$

Theorems 3, 5–9 of [3] readily yield the following statements.

Theorem 1. System of equations (3.1) is consistent.

Theorem 2. System (3.1) has a unique solution if and only if $\operatorname{rank} \| A' \ H \| = s$.

Theorem 3. $\lambda = 0$ if and only if $\operatorname{rank} A < \operatorname{rank} \| A' \ H \|$.

4. Theorem 4. Let the origin be an interior point of the domain E. There exists a vector φ_0 such that $\sup |\varepsilon' S\varphi_0| = 0$ if and only if

$$\operatorname{rank} S < \operatorname{rank} \| S' \ H \|$$

Proof. Necessity. If the vector φ_0 mentioned in the condition does exist, then $\varepsilon' S\varphi_0 = 0$ for any $\varepsilon \in E$. Since the direction of the vector ε is arbitrary, it follows that $S\varphi_0 = 0$, $S'S\varphi_0 = 0$. Applying Theorem 3 and setting $A = S$ and $\lambda = 0$ in (3.1), we find that $\operatorname{rank} S < \operatorname{rank} \| S' \ H \|$.

Sufficiency. Let the matrices S and $\| S' \ H \|$ be of different rank. This, by Theorems 1 and 3, means that the equations $S'S\varphi = 0$, $H'\varphi = 1$ are consistent. The vector φ obtained by solving this system (its first equation can be replaced by the equivalent equation $S\varphi = 0$) ensures the fulfillment of the equation $dV = 0$ for any ε.

5. In this section we consider the case where E is a nondegenerate ellipsoid in n-dimensional space with its center at the origin. The inclusion $\varepsilon \in E$ is specified by means of the formula

$$\varepsilon' K\varepsilon \leqslant 1$$

Here K is a symmetric positive-definite matrix. Let $\sqrt{K}\varepsilon = \xi$. Then

$$dV = \varepsilon' S\varphi = \xi' (\sqrt{K})^{-1} S\varphi$$

where ξ belongs to the unit sphere X.

If the vector φ is given, then the vector ξ which maximizes the quantity dV is a unit vector whose direction is the same as that of $(\sqrt{K})^{-1}S\varphi$,

$$\max_{\varepsilon\in E} |dV| = \max_{\xi\in X} \xi' (\sqrt{K})^{-1} S\varphi = \sqrt{\varphi' F\varphi} \qquad (F = S'K^{-1}S) \quad (5.1)$$

Next, we must find $\mu_0 = \min \sqrt{\varphi' F\varphi}$ for $\varphi \in \Phi$. It is clear that this problem differs from the stochastic one in the fact that K^{-1} and μ^2_0 are replaced by D and λ. The quantity μ_0 and the optimal values of φ can be determined by solving the system of equations

$$F\varphi = \mu_0^2 H, \qquad (H, \varphi) = 1 \quad (5.2)$$

Theorem 5. The optimal vector $\varphi \in \Phi$ is unique if and only if rank $\| S' \ H \| = s$. The proof follows from Theorem 2 with allowance for the fact that the rank of the matrix K is equal to n by virtue of its nonsingularity.

We see that the optimal controls for the stochastic and minimax variants of the problem coincide in the case where E is an ellipsoid. It is, of course, necessary to compare the cases where the matrices D and K^{-1} differ by a scalar factor only.

6. In this section we consider the case where E is a nondegenerate parallelepiped in n-dimensional space with its center at the origin. The points of such a parallelepiped can be expressed in parametric form,

$$\varepsilon = T\xi \quad (\xi \in \Xi = \{\xi : |\xi_i| \leqslant 1\})$$

Here T is a nonsingular matrix of order $n \times n$. Then

$$dV = \varepsilon' S\varphi = \xi' C\varphi \qquad (C = T'S)$$

For a fixed φ we have

$$\max_{\varepsilon\in E} \varepsilon' S\varphi = \max_{\xi\in\Xi} \xi' C\varphi = (\gamma, \operatorname{sign}\gamma) \qquad (\gamma = C\varphi \in \Gamma) \quad (6.1)$$

Here Γ is a linear manifold in $R^{(n)}$ which is the mapping of the domain Φ from $R^{(s)}$ into $R^{(n)}$ by means of the operator C ; sign γ is a vector with the components sign γ_i. We have thus reduced the problem to the determination of

$$\min_{\varphi\in\Phi} (C\varphi, \operatorname{sign} C\varphi) = \min_{\gamma\in\Gamma} (\gamma, \operatorname{sign}\gamma)$$

The coordinate axes contained in $R^{(n)}$ can be used to form 2^n combinations. Let us assign to each such combination a subspace which is a Cartesian product of the coordinate axes occurring in the given combination. This yields 2^n subspaces R^α, where α is an n-dimensional vector: $\alpha_i = 1$ if the ith axis occurs in the given combination; otherwise $\alpha_i = 0$. For example, the zeroth subspace for $R^{(2)}$ is $R^{\{0,0\}}$, the coordinate axes are $R^{\{1,0\}}$ and $R^{\{0,1\}}$ and the entire plane is $R^{\{1,1\}}$.

R^α contains $2^{\alpha'\alpha}$ orthants. We shall say that β is the direction vector of an orthant if $|\beta_i| = \alpha_i$ and if β belongs to the given orthant. We denote a closed orthant with the direction vector β by O^β. For example $O^{\{1,1\}}$ is the first orthant in $R^{(2)}$ and $O^{\{-1,1,0\}}$ is the second orthant in $R^{\{1,1,0\}} \subset R^{(3)}$.

Let P^α be the matrix of the orthogonal projection in $R^{( )}$ onto R^α, and let $G^\alpha = I - P^\alpha$, where I is an identity matrix. The distance from the point γ to R^α is $\sqrt{(G^\alpha\gamma, G^\alpha\gamma)}$, and the inclusion $\gamma \in R^\alpha$ is expressed by the equation $G^\alpha\gamma = 0$.

Let $\Gamma^\alpha = \Gamma \cap R^\alpha$, and let Φ^α be the set of those φ for which $\gamma = C\varphi \in \Gamma^\alpha$.

It is clear that the relation $\varphi \in \Phi^\alpha$ is fulfilled for those and only those φ which satisfy the system of equations

$$G^\alpha C\varphi = 0, \qquad (H,\ \varphi) = 1 \tag{6.2}$$

Let us investigate these equations.

Theorem 6. System (6.2) is consistent if and only if rank $(G^\alpha C) < \text{rank} \|C'G^\alpha\ H\|$.

The proof follows from Theorem 3. We need merely replace A by $G^\alpha C$ and note that

$$G^\alpha G^\alpha = G^\alpha$$

Theorem 7. System (6.2) has a unique solution if and only if

$$\text{rank } (G^\alpha C) < \text{rank } \|C'G^\alpha\ H\| = s$$

The proof of this theorem follows from Theorems 6 and 2.

Theorem 8. There exists a unique vector γ defined by system (6.2) and the equation $\gamma = C\varphi$ if and only if

$$\text{rank } (G^\alpha C) < \text{rank } \|C'G^\alpha\ H\| = \text{rank } \|C'\ H\|$$

Proof. Necessity. Let there exist a unique vector γ which satisfies the condition of the theorem. The inequality and the conditions of the theorem are then fulfilled. We set $\Delta\varphi = \varphi^{(1)} - \varphi^{(2)}$, where $\varphi^{(1)}$ and $\varphi^{(2)}$ are any solutions of (6.2). Then

$$G^\alpha C\Delta\varphi = 0, \quad (H,\ \Delta\varphi) = 0 \tag{6.3}$$

By virtue of the uniqueness of γ we have $C\Delta\varphi = 0$. This implies the validity of the system of equations

$$C\Delta\varphi = 0, \ (H,\ \Delta\varphi) = 0 \tag{6.4}$$

Since systems (6.3) and (6.4) are equivalent, the ranks of their matrices are equal.

Sufficiency. The consistency of Eqs. (6.2) and the existence of γ are self-evident. Since systems (6.3) and (6.4) have equal ranks by hypothesis, and since the former is a consequence of the latter, they are equivalent. This means that the vector φ is the same for all solutions γ of Eqs. (6.2).

Our problem consists in finding min $(\gamma, \text{sign } \gamma)$, where $\gamma \in \Gamma$. Let us consider the problem θ^α, i.e. the problem of finding min $(\gamma, \ \text{sign } \gamma)$, where $\gamma \in \Gamma^\alpha$, and then determining the corresponding values of γ and φ.

Theorems 6–8 enable us to determine whether the sets Γ^α and Φ^α are empty and whether they contain one or more than one element. The following cases are possible.

1) The sets Γ^α and Φ^α are empty. The problem θ^α is meaningless in this case.

2) The sets Φ^α and Γ^α contain one element each. These elements yield the solution of the problem θ^α.

3) The set Φ^α contains more than one element and Γ^α only one element γ.

The solution of the problem θ^α is given by this value of γ or by any $\varphi \in \Phi^\alpha$.

4) The sets Φ^α and Γ^α contain more than one element each.

Let us find the sets $\Gamma^{\alpha\beta} = \Gamma^\alpha \cap O^\beta$ for the α and $\beta \in B^\alpha = \{\beta : |\beta_i| = \alpha_i\}$ under consideration. Clearly, $\Gamma^\alpha = \bigcup_\beta \Gamma^{\alpha\beta}$, where β assumes all values from B^α. It is therefore sufficient to solve the problems $\theta^{\alpha\beta}$, which consist in finding min $(\gamma,\ \text{sign}\gamma)$, where $\gamma \in \Gamma^{\alpha\beta}$, for all $\beta \in B^\alpha$ and of choosing the smallest of the resulting minima.

For $\gamma \in O^\beta$ we have $\text{sign}\gamma = \beta$. Since the nonempty $\Gamma^{\alpha\beta}$ are closed convex, and since $(\beta,\ \gamma)$ for $\gamma \in \Gamma^{\alpha\beta}$ is a linear nonnegative function of the argument γ whose equivalent manifolds are bounded, it follows either that this function attains a minimum only on the boundary of $\Gamma^{\alpha\beta}$ or that it is constant on $\Gamma^{\alpha\beta}$.

Let A^α be the set of those α^* for which

$$R^{\alpha*} \subset R^\alpha, \qquad \dim R^{\alpha*} = \dim R^\alpha - 1$$

From now on we shall consider only those α^* for each α which belong to the set A^α.

Since the boundary of the nonempty $\Gamma^{\alpha\beta}$ consists of points belonging to some set $\Gamma^{\alpha*}$, it follows that instead of the problems $\theta^{\alpha\beta}$ we need merely consider the problems $\theta^{\alpha*}$ and then choose the smallest of the resulting minima.

If (γ,β) is constant on $\Gamma^{\alpha\beta}$ (for a fixed α this can happen for only one $\beta \in B^\alpha$), then the function $(\gamma, \operatorname{sign} \gamma)$ has the same value on $\Gamma^{\alpha*}$ as it does on $\Gamma^{\alpha\beta}$.

In this case the solution of the problem θ^α is given by any convex linear combination of any vectors φ (one from each $\Phi^{\alpha*}$). We can now readily construct an algorithm for solving the principal problem $\theta^{\{1,\ldots,1\}}$.

Let us describe the procedure Ψ as it applies to the problem θ^α. The first step is to determine which of Cases (1)–(4) applies to the problem θ^α.

If Cases (2), (3) apply, we include the corresponding vector in the set A.

If Case (4) applies, we include the corresponding vector α in the set Q.

If Case (1) applies, then: (a) for $\alpha \neq 0$ we include the coordinate axes occurring in the given R^α in the set Q_r, where $r = \dim R^\alpha$; (b) for $\alpha = 0$ we include all the coordinate axes in Q_0.

Let us first apply the procedure Ψ to all the θ^α where $\dim R^\alpha = \max\{0, n - s + 1\}$. If it turns out that $\alpha \in Q$, then we can apply the procedure Ψ to all the $\theta^{\alpha*}$, where $\alpha^* \in A^\alpha$. Having obtained all of the elements of some set Q_r, where $r \geqslant n - s + 1$, we use all of the axes occurring in Q_r to construct all of the possible sums of $r + 1$ terms and then apply the procedure Ψ to the corresponding θ^α. We apply the procedure Ψ to each problem θ^α not more than once. It is clear that the process just described is finite.

Let us determine $\mu_\alpha = (\gamma^\alpha, \operatorname{sign} \gamma^\alpha)$ where γ^α is a unique element from Γ^α) for all $\alpha \in A$. Let

$$A_0 = \{\alpha : \mu_\alpha = \min \mu_{\alpha^0}, \quad \alpha^0 \in A\}$$

The set of all optimal vectors φ coincides with the set of all convex linear combinations of any φ (one from each Φ^α), where $\alpha \in A_0$.

The final result takes the form

$$\mu_0 = \min_{\alpha\in A} \mu_\alpha, \quad \varphi_0 = \sum_{\alpha\in A_0} \lambda_\alpha\varphi^\alpha \qquad (\varphi^\alpha \in \Phi^\alpha, \lambda_\alpha \geqslant 0, \ \sum_{\alpha\in A_0} \lambda_\alpha = 1)$$

The above algorithm is generally a rapid means of arriving at our goal. In fact, if all of the minors of order $s - 1$ of the matrix C and of order s of the matrix $\|C' H\|$ are not equal to zero, then the sets Q and Q_r turn out to be empty, so that we need consider only those problems θ^α where $\dim R^\alpha = n - s + 1$, and Case (2) applies for all these θ^α. This requires the solution of C_n^{s-1} systems of s linear algebraic equations with s unknowns.

It would be interesting to compare the results of applying the vector φ optimal for the stochastic and minimax (ellipsoid) variants of the problem to the minimax (parallelepiped) variant, and vice versa. It is, of course, necessary to compare cases where the matrices D, K^{-1}, C differ by a scalar factor only. We can show that the result ($\max_\varepsilon |dV|$ or $\sqrt{\langle (dV)^2 \rangle}$) obtained in these cases exceeds the optimal result by a factor not larger than $\sqrt{n}$.

7. Examples. let

$$M = \begin{Vmatrix} 1 \\ 2 \\ 3 \end{Vmatrix}, \qquad \omega = 1, \qquad L = \begin{Vmatrix} 0 & 1 \\ 0 & 3 \\ 0 & 5 \end{Vmatrix}, \qquad H = \begin{Vmatrix} 1 \\ 2 \end{Vmatrix}$$

Then

$$S = MH' - \omega L' = \begin{Vmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{Vmatrix}$$

Let us consider two cases.

1) Let $E = E^e$ be a sphere of unit radius with its cenrer at the origin. Then

$$K = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \qquad F = S'S = \begin{Vmatrix} 14 & 6 \\ 6 & 3 \end{Vmatrix}$$

Equations (5.2) are as follows:

$$14\varphi_1 + 6\varphi_2 = \mu_0^2, \quad 6\varphi_1 + 3\varphi_2 = 2\mu_0^2, \quad \varphi_1 + 2\varphi_2 = 1$$

Their solution is

$$\varphi_0^e = \{-{}^9/_{35},\ {}^{22}/_{35}\}, \quad \mu_0^e = \sqrt{{}^6/_{35}}$$

2) Let $E = E^p$ be a cube all of whose vertices are equal to unity in absolute value. Here

$$T = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \qquad C = S, \qquad \operatorname{rank} \| C' \ \ H \| = s = 2$$

We apply the procedure Ψ to all of the θ^α, where $\dim R^\alpha = n - s + 1 = 2$. For all three such α we have

$$\operatorname{rank}(G^\alpha C) = 1, \qquad \operatorname{rank} \begin{Vmatrix} G^\alpha C \\ H' \end{Vmatrix} = \operatorname{rank} \begin{Vmatrix} C \\ H' \end{Vmatrix} = s = 2$$

System (5.2) is consistent by virtue of Theorems 6–8. The corresponding vectors φ and γ are unique, i.e. Case (2) applies. Let us construct system (6.2) for each of these α. The resulting equations enable us to determine φ and then $\gamma = C\varphi$ and $\mu = (\gamma, \operatorname{sign} \gamma)$. For $\alpha = \{1,1,0\}$ we have $\varphi = \{-{}^1/_5, {}^3/_5\}$, $\gamma = \{{}^2/_5, {}^1/_5, 0\}$, $\mu = {}^3/_5$. For $\alpha = \{1, 0, 1\}$ we have $\varphi = \{-{}^1/_3, {}^2/_3\}$, $\gamma = \{{}^1/_3, 0, -{}^1/_3\}$, $\mu = {}^2/_3$. For $\alpha = \{0, 1, 1\}$ we obtain $\varphi = \{-1, 1\}$, $\gamma = \{0, -1, -2\}$, $\mu = 3$. Comparison of the quantities μ shows that

$$\varphi_0^p = \{-{}^1/_5, {}^3/_5\}, \quad \mu_0^p = {}^3/_5$$

Applying to the sphere the vector φ optimal for the cube and making use of (5.1), we find that

$$\max |\varepsilon' S \varphi_0^p| = 1 / \sqrt{5} \qquad (\varepsilon \in E^e)$$

Applying to the cube the vector φ optimal for the sphere and making use of (6.1), we find that

$$\max |\varepsilon' S \varphi_0^e| = {}^{22}/_{35} \qquad (\varepsilon = \in E^p)$$

We also note that in the absence of the control under consideration

$$\max |dV| = \sqrt{14} \quad (\varepsilon \in E^e), \qquad \max |dV| = 6 \quad (\varepsilon \in E^p)$$

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BIBLIOGRAPHY

1. Ishlinskii, A. Iu., Internal Control of Ballistic Missiles. Moscow, "Nauka", 1968.
2. Drenick, R., The perturbation calculus in missile ballistics. J. Franklin Inst. Vol. 251, №4, 1951.
3. Cherenkov, A. P., Single-switching control for small perturbations. Teor. Veroiat. Prim. Vol. 9, №2, 1964.

4. Pshenichnyi, B. N., The dual method in extremal problems, I. Kibernetika №3, 1965.
5. Liusternik, L. A. and Sobolev, V. I., Elements of Functional Analysis. 2nd ed. Moscow, "Nauka, 1965.

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BOUNDARY VALUE PROBLEMS IN NONLOCAL THEORY OF ELASTICITY

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A. M. VAISMAN and I. A. KUNIN
(Novosibirsk)
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In recent years a large number of papers have been devoted to the development of various models of elastic media with microstructure. An analysis shows that in all these models there is some scale parameter l, which can characterize the discreteness, the long-range effectiveness, the scale of correlation, and so forth. The appropriate theories can be regarded as weakly or strongly nonlocal. The former are represented by the continuum theory of Cosserat, the couple-stress theory, the multipolar theory of elasticity, and so forth (e. g. see [1–4]). All these can be interpreted as the next approximation with respect to the usual (local) theory of elasticity. The parameter l in these cases must be considered as small.

Strongly nonlocal theories which do not assume the smallness of l, were examined in [5–7] (see also review [8]) for unbounded media.

In this paper boundary value problems of nonlocal theory are examined; the transition from exact to approximate models is investigated; a connection is established with boundary value problems of weakly nonlocal theories [9, 10] in the formulation of which the physical significance of boundary conditions was previously unclear.

The major portion of this work is devoted to one-dimensional problems. In Sect. 1, coupling conditions of two media with microstructure are examined, the analog to Green's formula is constructed, the fundamental boundary value problems and their equivalent integral equations are written down. In Sect. 2, the structure of the general solution of equations of motion for a homogeneous medium is examined. It is shown that the problem is reduced to the determination of roots of the energy operator in the complex plane of wave numbers. Green's function is constructed. Characteristic differences between the nonlocal and the classical theory are examined.

Section 3 is devoted to various approximate models and their regions of applicability. The long-wave approximation is compared to the approximation developed in this paper using first roots of the energy operator. The advantages of the latter approximation will be, the correct description of phenomena for which waves with the length of the order l are essential, the preservation of the principal terms of the asymptotics, and the possibility of correct approximate formulation of the boundary value problems. In Sect. 4, as an illustration the exact and approximate solutions of the fundamental problems are examined for the semi-bounded region.

In Sect. 5 some generalizations are presented for the case of a three-dimensional medium with central interaction.